

Sharply 3-transitive groups

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Abstract

We construct the first sharply 3-transitive groups not arising from a near field, i.e. point stabilizers have no nontrivial abelian normal subgroup.¹

1 Introduction

The finite sharply 2- and 3-transitive groups were classified by Zassenhaus in [Z1] and [Z2] in the 1930's and were shown to arise from so-called near-fields. They essentially look like the groups of affine linear transformations $x \mapsto ax + b$ or Moebius transformations $x \mapsto \frac{ax+b}{cx+d}$, respectively.

It remained an open problem whether the same holds for infinite sharply 2- and 3-transitive groups and much literature on this topic is available, see [RST] for background and more recent references. In [RST] the first construction of sharply 2-transitive groups without any nontrivial abelian normal subgroup is given.

However the question remained open whether the groups constructed there can be extended to groups acting sharply 3-transitively. We here use a different approach directly constructing sharply 3-transitive groups using partial group actions. These are the first known examples of sharply 3-transitive groups whose point stabilizers have no non-trivial abelian normal subgroup and thus do not arise from near-fields.

By results of Tits [Ti] and Hall [Ha] there are no infinite sharply k -transitive groups for $k \geq 4$, see e.g. [DM].

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2 The main theorem

For brevity we call an element of order 3 a 3-cycle and we say that a group action is 3-sharp if all 3-point stabilizers are trivial.

Theorem 2.1. *Let G_0 be a group in which all 3-cycles and all involutions, respectively, are conjugate and such that there exists a 3-cycle a and an involution t in G_0 with $\langle a, t \rangle \cong S_3$. Assume that G_0 acts on a set X_0 in such a way that*

1. *the action is 3-sharp;*
2. *the involution $t \in G_0$ fixes a unique point $x_0 \in X_0$;*
3. *the 3-cycle $a \in G_0$ is fixed point free;*
4. *we have $(x_0, x_0a, x_0a^2) = (x_0, x_0a, x_0at)$;*
5. *if $B = (x, y, z)$ is a triple in X_0 such that the setwise stabilizer of B in G_0 is isomorphic to S_3 , then there is some $g \in G_0$ with $Ag = B$ where $A = (x_0, x_0a, x_0a^2) = (x_0, x_0a, x_0at)$.*

Then we can extend G_0 to a sharply 3-transitive action of

$$G = ((\langle a \rangle \times F(U)) *_{\langle a \rangle} G_0 *_{\langle t \rangle} ((\langle t \rangle \times F(S)))) * F(R)$$

on a suitable set $Y \supset X_0$, where $F(R), F(S), F(U)$ are free groups on disjoint sets R, S, U with $|R|, |S|, |U| = \max\{|G_0|, \aleph_0\}$.

Remark 2.2. *Note that S_3 in its natural action on three elements satisfies the assumptions of Theorem 2.1 (as does $PGL(2, 2) \cong S_3$ acting as a subgroup of $PGL(2, 2^3)$ on the projective line over \mathbb{F}_{2^3} .)*

Therefore we have

Corollary 2.3. *There exist groups G acting sharply 3-transitively on some set X such that for $x \in X$ the point stabilizer G_x of x has no nontrivial abelian normal subgroup.*

Proof. Applying Theorem 2.1 to $G_0 = S_3 = \langle a \rangle \rtimes \langle t \rangle$ we see that no two distinct involutions in G commute. The point stabilizer G_x acts sharply 2-transitively on $X \setminus \{x\}$. Since the involutions in G_0 and hence (by conjugacy)

also in G have unique fixed points, the involutions of G_x in their action on $X \setminus \{x\}$ are fixed point free, so the sharply 2-transitive group G_x is said to have characteristic 2. By [Ne] (see also e.g. [BN], 11.46) a nontrivial abelian normal subgroup of G_x would have to consist of elements of order 2, each being the product of two involutions in G_x . From the construction of G , it can be seen that this is impossible in G . \square

3 The construction

In this section we prove Theorem 2.1, i.e. we construct G and its action on a set X from G_0 and X_0 . We continue to use the notation introduced in the previous section.

For the construction we use partial group actions as in [TZ], considered also by Rips and Segev. The construction proceeds by extending inductively both the group G_0 and the underlying set X_0 in such a way that the assumptions of Theorem 2.1 are preserved. This is done in two separate kinds of extensions: on the one hand we extend the group G_0 in order to make the action a bit more 3-transitive. On the other hand we extend the underlying set X_0 in order to let the extended group act, forcing us in turn to extend the group again etc. In the limit, the group G will be sharply 3-transitive on a set X containing X_0 .

Definition 3.1. *A partial action of G on a set X containing X_0 consists of an action of G_0 on X and partial actions of the generators in $S \cup R \cup U$ such that*

1. *for $s \in S$ we have $x_0s = x_0$ and if xs is defined for $x \in X \setminus \{x_0\}$, then so is $(xt)s$ and we have $(xt)s = (xs)t$.*
2. *for $u \in U, x \in X$, if xu is defined, then so are $(xa)u$ and $(xa^2)u$ and we have $(xa)u = (xu)a$ and $(xa^2)u = (xu)a^2$.*

We now define normal forms suitable for our purpose:

Normal forms 3.2. Any element of G can be written (not necessarily uniquely) as a reduced word in the generators $G_0 \setminus \{1\} \cup S \cup S^{-1} \cup R \cup R^{-1} \cup U \cup U^{-1}$ where we say that a word is reduced if there are no subwords of the form

- ff^{-1} for $f \in S \cup S^{-1} \cup R \cup R^{-1} \cup U \cup U^{-1}$,

- gh for $g, h \in G_0 \setminus \{1\}$;
- s_1ts_2 where $s_1, s_2 \in S \cup S^{-1}$;
- $ts_1 \cdots s_ng$ (or its inverse) where $s_i \in S \cup S^{-1}, i = 1, \dots, n, g \in G_0 \setminus \{1\}$;
- $u_1a^{\pm 1}u_2$ where $u_1, u_2 \in U \cup U^{-1}$;
- $a^{\pm 1}u_1 \cdots u_ng$ or its inverse where $u_i \in U \cup U^{-1}, i = 1, \dots, n, g \in G_0 \setminus \{1\}$.

The word w is called *cyclically reduced* if w and every cyclic permutation of w is reduced.

We say that for a word $w = s_1 \cdots s_n$ in $G_0 \setminus \{1\} \cup S \cup S^{-1} \cup R \cup R^{-1} \cup U \cup U^{-1}$, the element xw is *defined* for $x \in X$ if for all initial segments of w the action on x is defined, i.e. $xs_1, \dots, (\dots(xs_1)\dots)s_i, i \leq n$, are defined and we write $xw = (\dots(xs_1)\dots)s_n$.

Notice that for elements from G_0 the action on X is defined everywhere. Hence if w, w' are reduced words with $w = w'$ in G and xw is defined, then xw' is defined as well and $xw = xw'$. Thus the expression $xg = y$ makes sense for $g \in G, x, y \in X$.

If G acts partially on X , then there is a canonical partial action on the set of triples

$$(X)^3 = \{(x, y, z) \in X^3 \mid |\{x, y, z\}| = 3\}.$$

Terminology and notation For a triple $C = (x, y, z)$ we say that $g \in G$ *shifts* C if $Cg = (y, z, x)$ or $Cg = (z, x, y)$ and we say that $h \in G$ *flips* C if $Ch = (x, z, y)$ or a shift of this. Note that if there are such elements $g, h \in G$ then the setwise stabilizer of C in G is isomorphic to S_3 . We also say that an element $g \in G$ leaves a triple C *invariant* if Cg is defined and is equal to C as a set. In this case we call C a g -triple. We call a triple *free* if the only element leaving it invariant is the identity.

We call two triples C and C' *connected* if there is $w \in G$ such that Cw is defined and equals C' . If $w \in G$ leaves a triple C invariant, then writing $w = s_1 \cdots s_n$ in reduced form, the triples $(C, Cs_1, Cs_1s_2, \dots, Cw)$ form a cycle which we call *braided* if Cw is equal to C as a set, but not necessarily as an ordered triple.

Definition 3.3. *We call a partial action of G on X good (and say that G acts well on X) if*

1. the 3-cycle a acts without fixed point on X ;
 2. the involution t has a unique fixed point, namely $x_0 \in X$;
- and for all triples $C \in (X)^3$ and $g, h \in G$ the following holds:
3. $Cg = C$ implies $g = 1$.
 4. If h flips C , then h is a conjugate of t .
 5. If g shifts C , then g is a conjugate of a .
 6. If g shifts C and h flips C , then there is some $g' \in G$ such that $Cg' = A$ where $A = (x_0, x_0a, x_0a^2) = (x_0, x_0a, x_0at)$.

Note that the original action of G_0 on $X = X_0$ is good and that A is the unique triple of X_0 invariant under both a and t . In order to make the action 3-transitive it suffices to connect A to any other triple of the underlying set. Notice that since a does not fix any point, we have $(x, xa, xa^2) \in (X)^3$ for all $x \in X$.

In the remainder of the section we extend a good partial action in two ways: by letting free generators take the (distinguished) triple A to all a -, t - and free triples in order to eventually make the action 3-transitive, and by extending the domain of the partial action of a free generator in order to eventually make the action total. We start with the last one:

Lemma 3.4 (Extending the free generators). *Assume that G acts well on X and that for some $x \in X$ and $f \in S \cup S^{-1} \cup U \cup U^{-1} \cup R \cup R^{-1}$ the expression xf is not defined. (Then xtf is not defined if $f \in S \cup S^{-1}$ and xaf, xa^2f are not defined in case $f \in U \cup U^{-1}$.)*

Let $x'G_0 = \{x'g_0 \mid g_0 \in G_0\}$ be a set of new elements on which G_0 acts regularly and extend the partial operation of G to $X' = X \cup x'G_0$ by putting

1. $xf = x'$ if $f \in R \cup R^{-1}$;
2. $xf = x'$ and $(xt)f = x't$ (and $x_0f = x_0$) if $f \in S \cup S^{-1}$;
3. $xf = x'$, $(xa)f = x'a$ and $xa^2f = x'a^2$ if $f \in U \cup U^{-1}$.

Then G acts well on X' .

Proof. First observe that this clearly defines a partial action in the sense of Definition 3.1 and that the actions of a and t still satisfy conditions 1. and 2. of Definition 3.3.

For the remaining conditions of 3.3 it suffices to prove that if a cyclically reduced word w leaves a triple C in X' invariant, then $w \in G_0$ (and hence w is conjugate to a or t by assumption) or C and the (possibly braided) cycle described by w applied to C are contained in X . Since the previous action was good, this is enough.

Suppose otherwise: let $w \in G \setminus G_0$ be cyclically reduced (in the sense of 3.2) leaving the triple C invariant. Assume that at least one triple of the cycle given by applying w to C does not belong to X .

First assume that there is a triple in the cycle which does not belong to X , but both its neighbours do. This easily implies that a cyclic permutation of w contains the subword ff^{-1} as f is the only element taking a triple from X to a triple not entirely belonging to X . So w is not cyclically reduced, a contradiction.

Next assume that there are two neighbouring triples C_1, C_2 in the cycle which do not belong to X . Then by the properties of a cyclically reduced word and the definition of X' , C_1 and C_2 are connected by an element $g \in G_0 \setminus 1$. So the cycle contains a segment (B, C_1, C_2, D) where the triples B, D are contained in X and necessarily $Bf = C_1, C_1g = C_2$ and $C_2f^{-1} = D$.

Then $f \notin R \cup R^{-1}$ as G_0 acts regularly on $x'G_0$.

If $f \in S \cup S^{-1}$, we must have $g = t$ since on $x'G_0$ the element f^{-1} is only defined on $x't$. So a cyclic permutation of w contains the subword $f \cdot t \cdot f^{-1}$, a contradiction.

Similarly, if $f \in U \cup U^{-1}$, we have $g = a^{\pm 1}$ and a cyclic permutation of w contains the subword $f \cdot a^{\pm 1} \cdot f^{-1}$, again a contradiction.

This shows that if a triple C becomes invariant under some $g \in G$ under the extended action, this is induced by conjugation under the previous action. Hence Condition 4. of 3.3 is preserved as well. \square

We next show how to extend the group action in order to connect the unique triple $A = (x_0, x_0a, x_0a^2)$ with $x_0at = x_0a^2$ to a triple B where B is either an a -triple, a t -triple or a free triple:

Lemma 3.5 (Connecting A to other triples). *Assume that G acts well on X , let $A = (x_0, x_0a, x_0a^2) = (x_0, x_0a, x_0at)$ be as before and let B be an a -, t - or a free triple for which there is no $g \in G$ with $Ag = B$. Let $f \in R \cup S \cup U$ be an element which does not yet act anywhere with*

1. $f \in U$ if B is an a -triple;
2. $f \in S$ if B is a t -triple;
3. $f \in R$ if B is a free triple

Extend the action by setting $Af = B$. Then this action of G on X is again good.

Proof. Again it is clear that this defines a partial action in the sense of Definition 3.1 and that 1. and 2. of Definition 3.3 continue to hold. Also note that since the setwise stabilizer of A in G is S_3 the assumptions imply that there is no $w \in G$ not containing f, f^{-1} taking A to B as a set.

To prove the remaining conditions of 3.3 let w be a cyclically reduced word (in the sense of 3.2) leaving some triple $C \in (X)^3$ invariant. Since the previous action was good, it suffices to show that w does not contain f or f^{-1} . Suppose otherwise. Then by cyclically permuting w and taking inverses we may assume that $w = f \cdot w'$. So w stabilizes A and w' takes B to A as a set. By assumption on A, B the subword w' must contain f . Hence we may write $w' = v' \cdot f^\epsilon v$ for some subword v' not containing f or f^{-1} . We distinguish two cases:

1. $\epsilon = 1$. Then v' takes B to A as a set as f is only defined on A . Since v' does not contain f, f^{-1} , this contradicts the assumption on A, B .
2. $\epsilon = -1$. Then v' leaves B invariant. Since the previous action was good, we either have $v' = t$ and $f \in S$ or $v' = a^{\pm 1}$ and $f \in U$ (according to whether B is an a - or a t -triple). In either case v' commutes with f , contradicting the assumption that w be cyclically reduced.

□

Corollary 3.6. *Assume that G acts well on X with $|X| \leq \max\{\aleph_0, |G|\}$ and there are sufficiently many elements of R, S and U whose action is not yet defined anywhere. Then we can extend the partial action of G on X to a sharply 3-transitive (total) action on some appropriate superset Y .*

Proof. Fix the unique a -triple $A = (x_0, x_0a, x_0a^2) = (x_0, x_0a, x_0at)$ in X_0 . Using the previous lemmas we define a set Y and a 3-sharp action of G on Y with the following properties:

1. all a -triples are connected to A ;

2. all t -triples are connected to A ;
3. any triple (x, y, z) can be shifted by an element of G .

The last property can be achieved using Lemma 3.5: suppose $B = (x, y, z)$ cannot be shifted by an element of G at a certain stage of the construction. Then B is a free triple: otherwise an element of G leaving B invariant would have to be an involution $t' \in G$. Since $t' = hth^{-1}$ for some $h \in G$, the triple Bh is a t -triple and hence connected to A , making B shiftable as well.

Thus, B is a free triple at that stage, and we later extend the action by putting $Ar = B$ for some $r \in R$. Then B can be shifted by a^r .

This easily implies that the action of G on Y is sharply 3-transitive, i.e. all triples are connected to A : let B be a triple and $g \in G$ shift B . Then we have $g = hah^{-1}$ for some $h \in G$, so Bh is an a -triple and whence connected to A . \square

This concludes the proof of Theorem 2.1.

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